ME 4555 – Lecture 30 – Bode plots, part II

Let's look at the frequency response of a 2nd order system (underdamped):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow G(j\omega) = \frac{\omega_n^2}{-\omega^2 + \omega_n^2 + 2\zeta\omega\cdot\omega_n j}$$

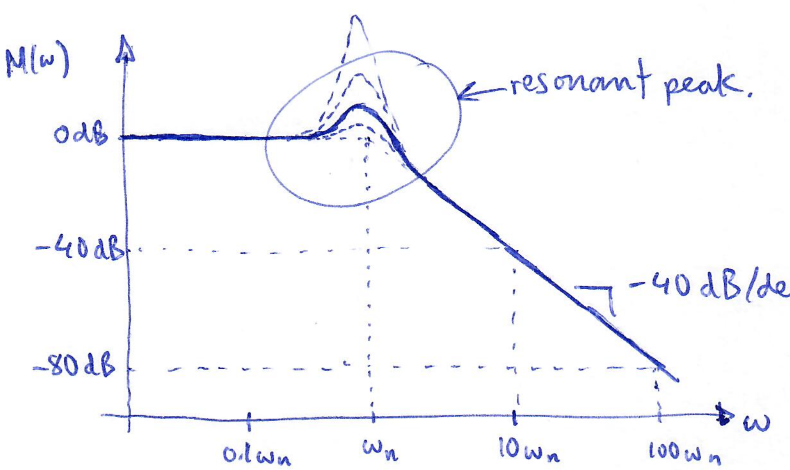
Rewrite as: $G(j\omega) = \dfrac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)j}$

If $\omega_n \ll \omega$ (high frequency), $G(j\omega) \approx -\dfrac{1}{\left(\frac{\omega}{\omega_n}\right)^2}$, therefore,

$$\begin{cases} M_{dB}(\omega) = 20\log_{10}|G(j\omega)| = -40\log_{10}\left(\frac{\omega}{\omega_n}\right) & \text{(slope of } -40 \text{ dB/decade)} \\ \phi(\omega) = -\pi = -180^\circ & \text{(since it's a negative number, real).} \end{cases}$$

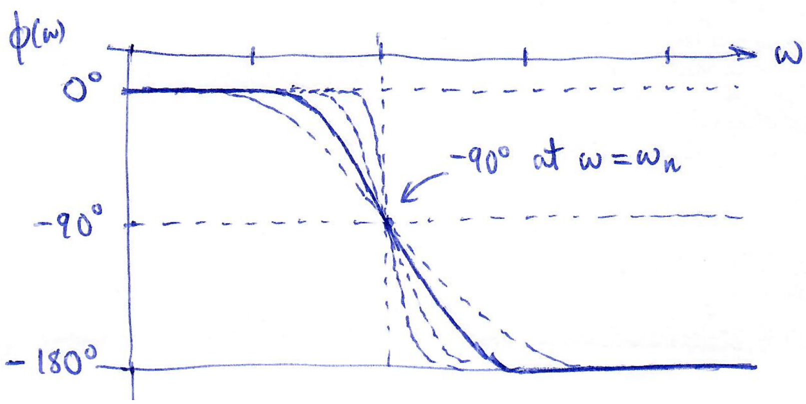
If $\omega_n \gg \omega$ (low frequency), $G(j\omega) \approx 1$, therefore

$M_{dB} = 0$ dB, $\phi = 0°$. (i.e. gain of 1). Here is the Bode plot:



for ζ closer to zero (less damping), peak is higher.

for $\zeta \geq \frac{1}{\sqrt{2}} \approx 0.707$, there is no peak.



for ζ closer to zero (less damping), transition is sharper

☆ verify this with the interactive demo!

We can derive the properties of the peak in $M(\omega)$:

$$G(j\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)j} \Rightarrow M(\omega) = |G(j\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

peak occurs when $\frac{d}{d\omega}M(\omega) = 0$, which is when $\frac{d}{d\omega}\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right] = 0$

evaluating the derivative + simplifying, this leads to $4\left(\frac{\omega}{\omega_n}\right)\left(\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta^2 - 1\right) = 0$

If $2\zeta^2 - 1 > 0$, i.e. $\zeta \geq \frac{1}{\sqrt{2}}$, the only solution is $\omega = 0$, so there is no resonant peak.

If $2\zeta^2 - 1 \leq 0$, i.e. $0 \leq \zeta < \frac{1}{\sqrt{2}}$, there is a peak at $\frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2}$

we call this the <u>peak frequency</u> $\boxed{\omega_p = \omega_n\sqrt{1-2\zeta^2}.}$

Note that $\omega_p < \omega_d < \omega_n$.

$\omega_n\sqrt{1-2\zeta^2}$ (peak freq) $\qquad$ $\omega_n\sqrt{1-\zeta^2}$ (damped freq) $\qquad$ (natural freq).

Peak magnitude is $M(\omega_p) = \dfrac{1}{\sqrt{(1-(1-2\zeta^2))^2 + 4\zeta^2(1-2\zeta^2)}} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

When ζ is small (say $\zeta < 0.5$), $M(\omega_p) \approx \frac{1}{2\zeta}$.

So the peak height in dB is $-20\log_{10}(2\zeta)$.

For example, if $\zeta = 0.05$, $M_{dB}(\omega_p) \approx -20\log_{10}(0.1) \approx$ **20 dB**

if $\zeta = 0.25$, $M_{dB}(\omega_p) \approx -20\log_{10}(0.5) \approx 6$ dB.

As $\zeta \to 0$, $M(\omega_p) \to \infty$.

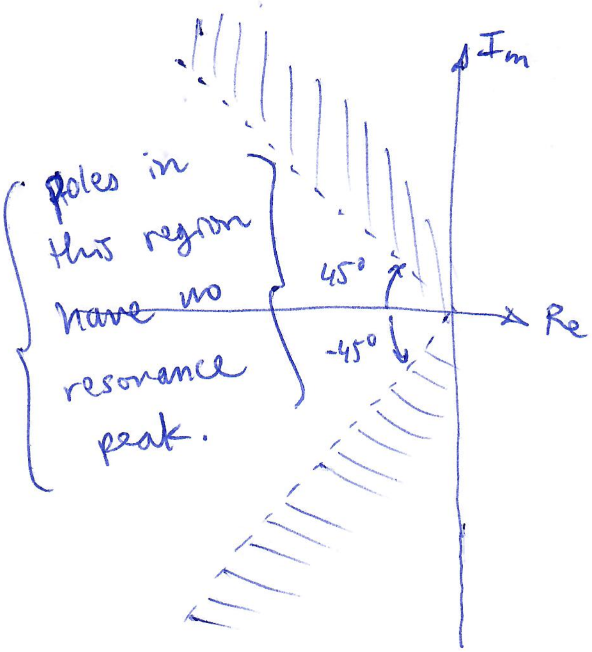
Both 1^{st} and 2^{nd} order systems act as low-pass filters

1^{st} order: $\dfrac{1}{\tau s + 1} \Rightarrow$ corner at $\omega_c = \dfrac{1}{\tau}$, -20 dB/decade for $\omega \gg \omega_c$.

2^{nd} order: $\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow$ corner at ω_n, -40 dB/decade for $\omega \gg \omega_c$.

Since $\zeta < \frac{1}{\sqrt{2}} \approx 0.707$ has a magnitude peak (resonance), it is often desirable to achieve $\zeta = \frac{1}{\sqrt{2}}$ since it is the least amount of damping required so that there is no peak on the Bode plot.

In the complex plane, $\zeta = \frac{1}{\sqrt{2}}$ corresponds to a pole angle of $45°$.



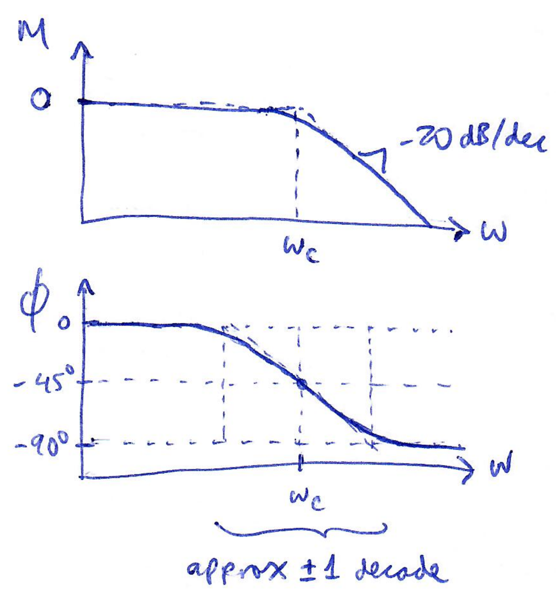
A low-pass filter with $\zeta = \frac{1}{\sqrt{2}}$ (2nd order) is called a <u>Butterworth filter</u>, and it has a transfer function of the form:

$$G(s) = \frac{\omega_n^2}{s^2 + \sqrt{2}\,\omega_n s + \omega_n^2}$$

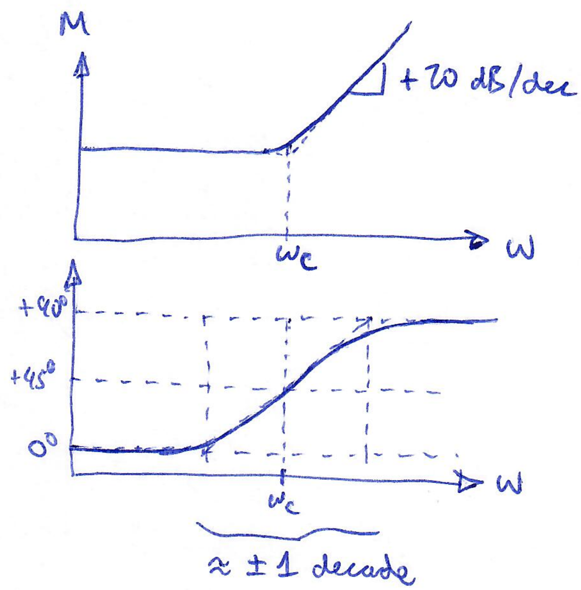
Sketching Bode plots.

Since magnitude is on a log-scale, we can sketch the Bode plots of each pole + zero separately and just add them up!

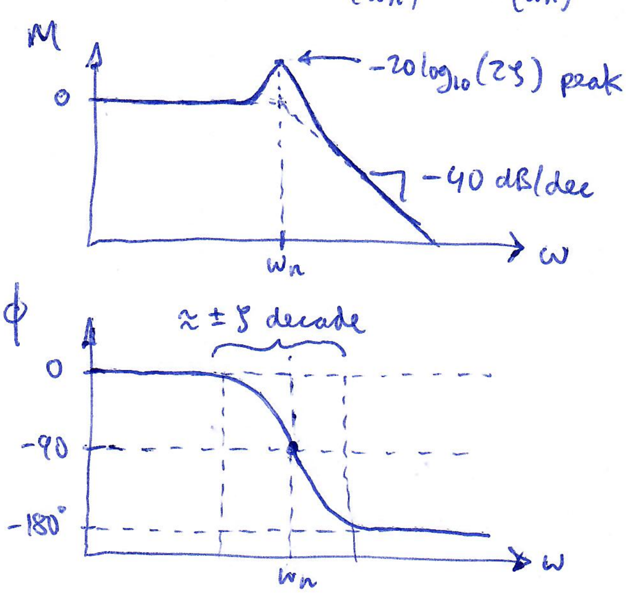
Real pole: $\dfrac{1}{\frac{s}{\omega_c}+1}$



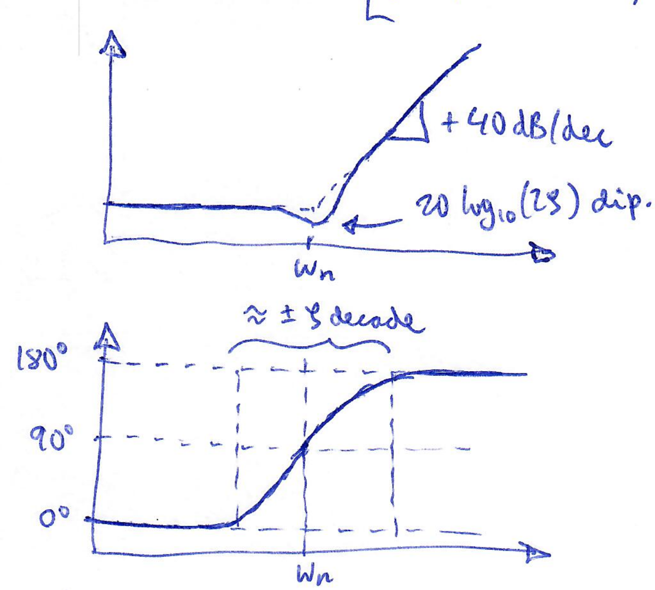
Real zero: $\left(\frac{s}{\omega_c}+1\right)$



Complex pole: $\dfrac{1}{\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1}$



Complex zero $\left[\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)+1\right]$.



Gain: K,
$$\begin{cases} M = 20\log_{10}|K| \\ \phi = 0^\circ \text{ if } K>0, \quad 180^\circ \text{ if } K<0 \end{cases}$$

Example $G(s) = \dfrac{100(s+1)}{s^2 + 110s + 1000}$. Sketch the Bode plot.

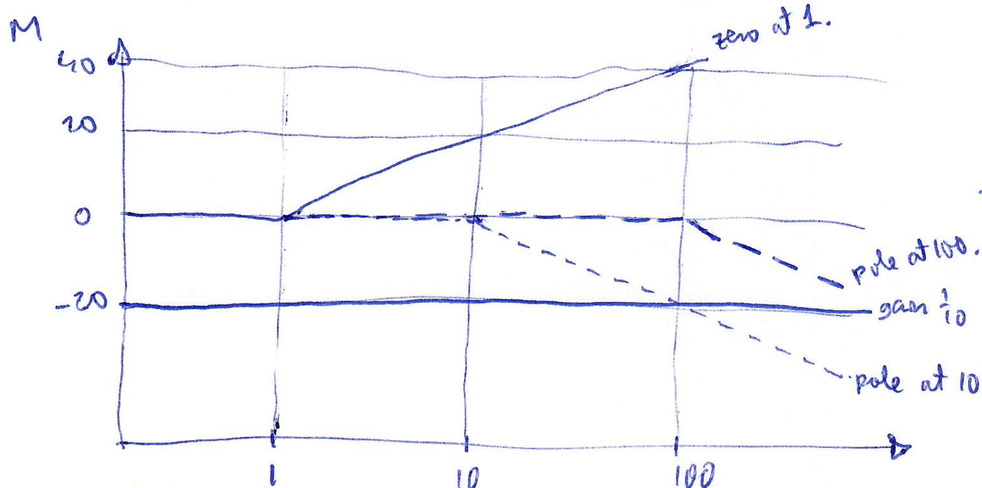
1) put into corner frequency form:

$$G(s) = \frac{100(s+1)}{(s+10)(s+100)} = \frac{1}{10} \cdot \frac{(s+1)}{\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$

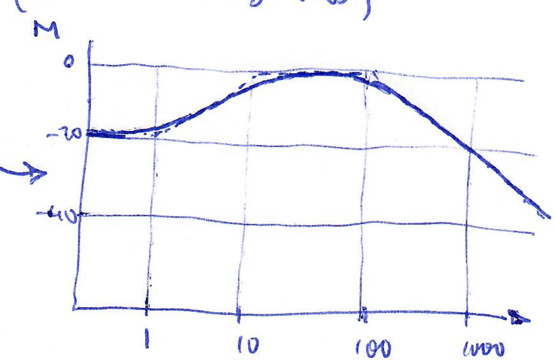
2) decompose: we have:

gain of $\frac{1}{10}$, zero at $w_c = 1$, pole at $w_c = 10$, pole at $w_c = 100$.

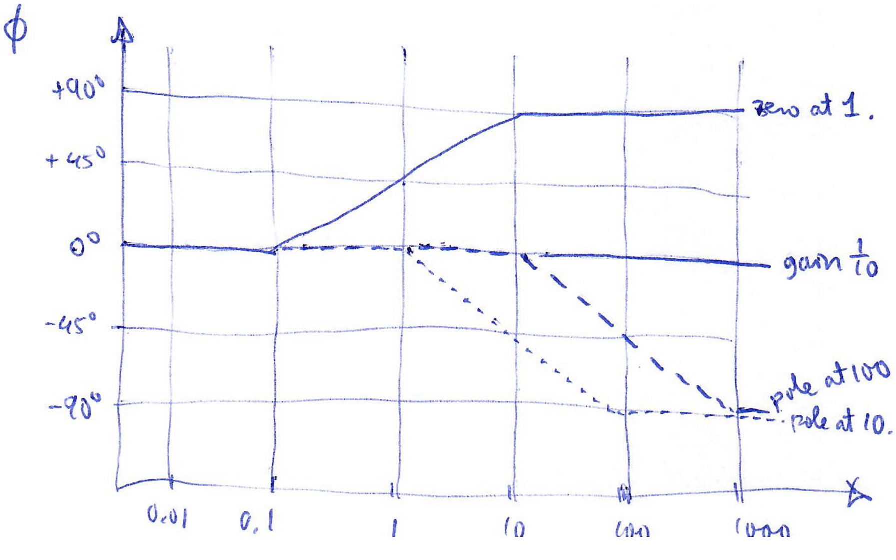
3) Magnitude plot:



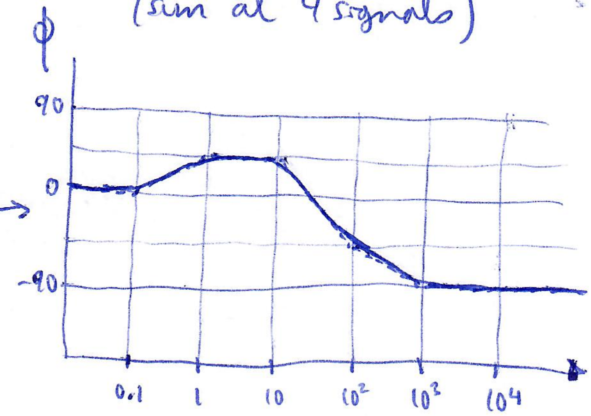
(sum all 4 signals)



4) Phase plot:



(sum al 4 signals)

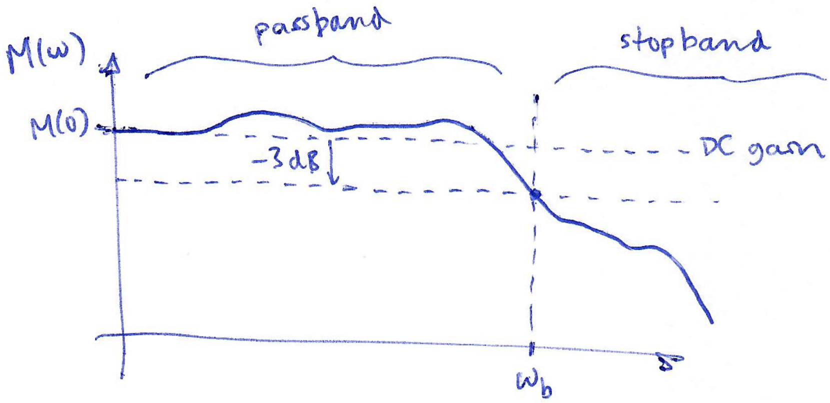


Most real-world systems act roughly like low-pass filters. (this is a good thing!)

When $\omega \to 0$, frequency response is $G(0)$. Recall this is the same as the steady-state response to a step input by FVT.
DC gain!

when $\omega \to \infty$, once we've exceeded all corner frequencies, the roll-off will occur at a rate of $-20\underbrace{(n-m)}_{\text{number of poles minus number of zeros}}$ dB/decade.

The point where the Bode gain plot crosses $|G(0)| - 3\text{dB}$ is called the **bandwidth** (ω_b) of the system.
i.e. frequency



Why -3dB? because a drop of 3 dB is $\frac{1}{\sqrt{2}}$ drop in amplitude, i.e. $\frac{1}{2}$ drop in power. It's where the system attenuates by 50% power.

For a 1^{st} order system, $\omega_b = \omega_c = \frac{1}{\tau}$.

For a 2^{nd} order system, $\omega_b = \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$

and when $0 < \zeta < \frac{1}{\sqrt{2}}$, we have: $\underset{\text{peak}}{\omega_p} < \underset{\text{damped}}{\omega_d} < \underset{\text{natural}}{\omega_n} < \underset{\text{bandwidth}}{\omega_b}$

Summary :

- a real pole causes a -20 dB/dec roll-off at $\omega \gg \omega_c$. ("low pass filter")
  also, a real pole causes $-90°$ of phase offset (phase lag)

- a real zero causes a $+20$ dB/dec gain at $\omega \gg \omega_c$.
  also, a real zero causes $+90°$ of phase offset (phase lead).

- a complex pair of poles has the same asymptotic effect as two real poles : -40 dB/dec roll-off and $-180°$ of phase offset.
  Except there will be a resonant peak of $\approx -20\log_{10} 2\zeta$ dB if $\zeta < \frac{1}{\sqrt{2}}$. No peak if $\zeta \geq \frac{1}{\sqrt{2}}$.

- Same goes for the complex pair of zeros (acts like two real zeros asymptotically).

- Most real-world systems behave like low-pass filters.
  ω_b is the frequency where the magnitude drops by 3 dB from the DC gain value (i.e. $\frac{1}{\sqrt{2}}$ drop in magnitude).
  The pass band is $\omega < \omega_b$ (frequencies below bandwidth)
  The stopband is $\omega > \omega_b$ (frequencies above the bandwidth).